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On a class of non-interpolating solutions of the many-anyon problem

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Abstract. In the many-anyon problem in two space dimensions, irregular but square integrable solutions of the Schrödinger equation may exist. A class of such solutions is constructed for anyons confined in a harmonic oscillator. It is shown that these may have lower energies than the usual regular solutions, but they do not exist throughout the range between the bosonic and fermionic limits, and as such do not interpolate continuously.

For a particle moving in a non-singular central potential, the Schrödinger equation demands that the wavefunction for a given angular momentum l scales as r^l or $r^{-(l+1)}$ as $r \to 0$. The irregular solution in bound state problems is normalizable only for $l = 0$. But this is not really a solution [1] of the Schrödinger equation at $r = 0$, since $\nabla^2(1/r)$ gives rise to a delta function. For this reason, the irregular 'solution' is excluded from the spectrum of physical states even though the corresponding eigenvalue may be lower than the physical ground state (as happens in the case of the three dimensional oscillator, for example).

In this paper we investigate the properties of such irregular solutions in two space dimensions in the presence of the statistical (anyonic) interaction [2, 3]. To begin with, we note that in two dimensions the wavefunction of two particles with the relative coordinate r scales as $r^{|l|}$ or $r^{-|l|}$ as $r \to 0$. For $l = 0$ both states coincide and there is no irregular solution, while for $|l| \geqslant 1$ the irregular solution is not normalizable. However if the 'angular momentum' is fractional (and less than one) as in the case of fractional statistics [2, 3], we would obtain perfectly valid solutions of the Schrödinger equation without the delta function anomalies. These irregular solutions have the property that they do not continuously interpolate in energy between the bose and the fermi limits as do the solutions which are regular. As we shall show later, the class of non-interpolating solutions not only includes the irregular solutions but also, in special cases, the regular solutions. It has also been realized [2] that while considering the exchange of two identical particles in a plane, the overlapping point $r = 0$ should be excluded from the configuration space. It is the winding around this special point $r = 0$ that leads to fractional statistics. One may suspect, therefore,

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that while considering two bosons with the anyonic interaction in two dimensions, the irregular but normalizable solutions may have some significance. In this paper, however, we argue to the contrary on the basis that these solutions do not interpolate continuously between the bose and fermi limits.

We show that even for the N-boson problem confined in a harmonic oscillator potential and interacting via the statistical interaction, there exists a class of normalizable irregular exact solutions. To this end, we shall follow the methods used earlier to study the regular solutions of the N-anyon problem [4, 5] . Unless otherwise stated, we assume the states to be bosonic. The N-particle Hamiltonian is conveniently written as ($\hbar = 1$)

$$H = \left[\frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N} r_i^2 - \frac{\alpha}{m}\sum_{j>i=1}^{N}\frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{i\neq j,k}^{N}\frac{\boldsymbol{r}_{ij}\cdot\boldsymbol{r}_{ik}}{r_{ij}^2 r_{ik}^2}\right] \tag{1}$$

where

$$\ell_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j)\times(\boldsymbol{p}_i - \boldsymbol{p}_j).$$

The relative angular momentum l_{ij} may be regarded as a scalar in two dimensions. Hereafter we regard all distances as dimensionless measured in units of $1/\sqrt{m\omega}$. Equivalently we set $m = \omega = 1$. In the above the parameter α ($0 \leqslant \alpha \leqslant 1$) denotes the strength of the statistical interaction and $\alpha = 0$ corresponds to the non-interacting bosonic limit. Note that the statistical interaction is independent of the centre of mass. Furthermore, the particles indices j and k can be equal in the last term.

For analytic manipulations it is more convenient to use the complex coordinates, $z_i = x_i + \mathrm{i}y_i$, for solving the eigenvalue equation, $H\psi = E\psi$. Note that ψ is symmetric under the exchange of any two particles, and E includes the energy of the centre of mass motion. It is of further advantage to make the following transformation [4–10]

$$\psi(z_i, \bar{z}_i) = \prod_{i<j} |\, z_{ij}\,|^{\beta}\, \eta(z_{ij}, \bar{z}_{ij}) \exp\left(-\frac{1}{2N}\sum_{i<j} |\, z_{ij}\,|^2\right)\phi_{\mathrm{CM}}(R) \tag{2}$$

where

$$z_{ij} = z_i - z_j \qquad R = \sum_{i=1}^{N} r_i/\sqrt{N}$$

and the overbars denote complex-conjugates. Substituting the expression for ψ in the eigenvalue equation, we obtain an eigenvalue equation in terms of η which is purely a function of the relative coordinates. The new eigenvalue equation is then given by

$$\tilde{H}\eta(z_{ij}, \bar{z}_{ij}) = \tilde{E}\eta(z_{ij}, \bar{z}_{ij}) \tag{3}$$

$$\tilde{E} = \left[E_{\mathrm{rel}} - (N-1) - \beta\frac{N(N-1)}{2}\right]. \tag{4}$$

Here $E_{\rm rel} = E - E_{\rm CM}$, with the centre of mass energy subtracted out. Since the statistical interaction is independent of the centre of mass motion, it is convenient to analyse the eigenvalues and solutions in terms of the relative motion alone. It should be remembered that the centre of mass wavefunction $\phi_{\rm CM}$ is always regular in two space dimensions. The reduced Hamiltonian is given by

$$\tilde{H} = \left[-2\partial_i\bar{\partial}_i + z_i\partial_i + \bar{z}_i\bar{\partial}_i - (\alpha+\beta)\sum_{i<j}\frac{\partial_{ij}}{\bar{z}_{ij}} \right. \\ \left. + (\alpha-\beta)\sum_{i<j}\frac{\bar{\partial}_{ij}}{z_{ij}} + \frac{\alpha^2-\beta^2}{2}\sum_{i\neq j,k}\frac{1}{\bar{z}_{ij}z_{ik}} \right]. \tag{5}$$

Here we use the notation $\partial_i = \partial/\partial z_i$, $\partial_{ij} = \partial_i - \partial_j$. Note that with the choice

$$\alpha^2 - \beta^2 = 0 \qquad \beta = \pm\alpha \tag{6}$$

the reduced Hamiltonian $\tilde{H}$ contains no three-body term, and is linear in α. So long as η is non-singular, regular solutions result from the choice $\alpha = \beta$ [4]. Here we concentrate on the second possibility, $\alpha = -\beta$ in the above equation, which may lead to irregular solutions when suitable conditions are imposed on the solution η. With the choice $\beta = -\alpha$, $\tilde{H}$ reduces to the form

$$\tilde{H} = \left[-2\partial_i\bar{\partial}_i + z_i\partial_i + \bar{z}_i\bar{\partial}_i + 2\alpha\sum_{i<j}\frac{\bar{\partial}_{ij}}{z_{ij}} \right]. \tag{7}$$

The lowest eigenvalue of equation (3), may be found by simply setting η to be a constant independent of the z's. Then $\tilde{H}\eta = 0$ identically. As a result $\tilde{E} = 0$, and it follows from equation (4) that

$$E_{\rm rel} = (N-1) - \alpha\frac{N(N-1)}{2}. \tag{8}$$

The corresponding wavefunction with angular momentum $L = 0$ is given by

$$\psi_{\rm rel} = \prod_{i<j} |\, r_{ij}\,|^{-\alpha} \exp\left(-\frac{1}{2N}\sum_{i<j} |\, r_{ij}\,|^2 \right). \tag{9}$$

Obviously the energy of the lowest irregular solution decreases linearly with α, in contrast to the regular ground state which has the energy eigenvalue $E_{\rm rel} = (N-1) + \alpha N(N-1)/2$. (The special case of $N = 2$, in the context of irregular solutions, has also been examined recently [11].)

Even though the wavefunction $\psi_{\rm rel}$ given by equation (9) is an exact N-body solution for all α, the states themselves are not normalizable for all α. Imposing the normalizability condition on $\psi_{\rm rel}$ given above, we have, by dimension counting

$$(2N-3) - \alpha N(N-1) > -1$$

where the first factor is the dimension of the measure and the second factor arises from $|\psi_{\rm rel}|^2$. In terms of α we have

$$\alpha < 2/N. \tag{10}$$

Obviously there is no continuous interpolation in energy from the ideal bose to the ideal fermi limit of the spectrum since $\alpha < 1$ (ideal fermi limit) for all N. It is in this sense that we refer to this and similar states, as the non-interpolating states. The known regular states have the property that they continuously interpolate between the bose and fermi limits.

A tower of exact solutions with $L = 0$ may be constructed through the radial excitations of this ground state. To this end we define a set of $(N-1)$ relative (Jacobi) coordinates [4]

$$\rho_i = \left[\frac{1}{\sqrt{i(i+1)}}\sum_{k=1}^{i} r_k - \sqrt{\frac{i}{i+1}}r_{i+1}\right] \qquad i = 1, \ldots, N-1. \tag{11}$$

We choose η of equation (3) to be a polynomial of the form

$$\eta(t) = \sum_{k=0}^{n} a_k t^k \qquad t = \sum_{i=1}^{N-1} \rho_i^2. \tag{12}$$

When η is a solution of the differential equation, the coefficients a_k are determined by the recursion relation and the normalization condition. The tower of eigenvalues is then given by the expression

$$E_{\rm rel} = 2n + (N-1) - \alpha\frac{N(N-1)}{2}. \tag{13}$$

The normalization condition, equation (10), remains unaltered since as t tends to zero, η tends to a constant.

The above discussion was intended to point out the existence of irregular solutions whose ground state energy is lower than the regular ground state energy for all α as given by equation (10). It is also important to point out that, by construction, these irregular states exist only when $\alpha \neq 0$. We now extend the method used above to obtain non-interpolating solutions for $L \neq 0$. It is, however, not our intention here to find the class of all non-interpolating solutions but merely to point out peculiarities that may occur when $L \neq 0$. In fact, when $L \neq 0$, it is possible to find regular solutions which are of the non-interpolating type. Consider, for example, a class of solutions of the form

$$\eta = \prod_{i<j}(z_{ij})^{l_{ij}} g(t) \qquad g(t) = \sum_{k=0}^{n} a_k t^k. \tag{14}$$

Since η is single-valued, l_{ij} must be a integer, and furthermore l_{ij} is an even integer for symmetric states. Alternatively, by choosing l_{ij} to be odd integers we could form a fermionic set of states. The total angular momentum of the state $L = \sum_{i<j} l_{ij}$ is also an integer . The eigenvalues are now given by

$$E_{\rm rel} = 2n + L + (N-1) - \alpha\frac{N(N-1)}{2}. \tag{15}$$

The normalization condition on the relative wavefunction yields the condition

$$\alpha < \frac{2(N-1+L)}{N(N-1)} \tag{16}$$

which should be satisfied by any admissible solution in the Hilbert space. Here L can take positive or negative integer values such that $0 \leqslant \alpha \leqslant 1$. Using the above condition we may now obtain a classification of various $L \neq 0$ states. Complete interpolation would require the domain of α to include the point $\alpha = 1$. This implies $L > (N-1)(N-2)/2$, and solutions which satisfy this criterion are already known in the literature [5, 7]. The non-interpolating states have $L \leqslant (N-1)(N-2)/2$, which, interestingly enough, have both regular and irregular solutions. These non-interpolating solutions remain regular in the range

$$0 \leqslant \alpha \leqslant 2L/N(N-1)$$

and irregular in the range

$$2L/N(N-1) \leqslant \alpha \leqslant 2(N-1+L)/N(N-1).$$

For $L \leqslant 0$ the solutions are necessarily irregular.

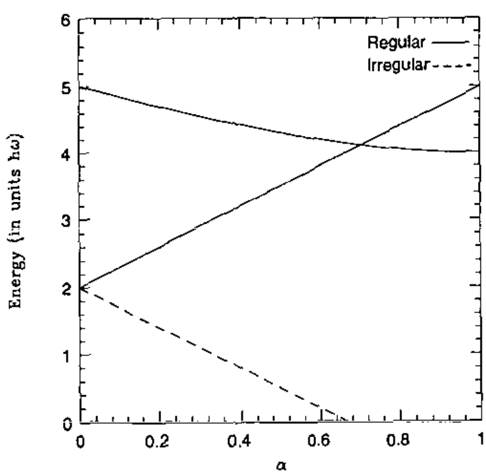

Figure 1. Low lying regular and irregular solutions as a function of α. The linear solutions are exact, while the nonlinear interpolation was obtained numerically. The irregular solutions do not exist for $\alpha \geqslant 2/3$. Note that $\alpha = 0$ is the bosonic end and $\alpha = 1$ is the fermionic end of the spectrum.

To summarize, the normalizability conditions (10) and (16) imply that the irregular solutions exist only in some limited range of α that shrinks as the number of particles N increases. Consequently even for $L = 0$, there is no continuous interpolation between the bose and fermi limits. A concrete example of such behaviour is shown in figure 1 for three anyons confined in an oscillator potential. The numerical calculations for the low-lying regular interpolating states are well known [12]. We also show in the same figure the lowest $L = 0$ irregular state from the bosonic end that exists only for $\alpha < 2/3$. The physical requirement that states which exhibit fractional statistics should interpolate continuously would exclude these partially interpolating, irregular (and regular when $|L| \neq 0$) solutions. Though the eigenvalue does not demand this criterion for solutions, this has to be imposed as an additional physical requirement for non-zero α. It is therefore of utmost importance to be aware of these solutions in numerical calculations of N-anyon systems (either on a lattice or in the presence of some confinement potential) which may yield lower energies, but should be excluded by the physical criterion.

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